Tutorial 8 for MATH 2020A (2024 Fall)

1. Consider the following differential form ω defined in its natural domain,

$$\omega = (ay^2 + 2czx)\,\mathrm{d}x + y(bx + cz)\,\mathrm{d}y + (ay^2 + cx^2)\,\mathrm{d}z.$$

How are the constants a, b and c related if ω is exact?

> **Solution:** $b = 2a$ and $c = 2a$.

2. Consider the following differential form

$$\omega = \left(\ln x + \sec^2(x+y)\right)\,\mathrm{d}x + \left(\sec^2(x+y) + \frac{y}{y^2+z^2}\right)\,\mathrm{d}y + \frac{z}{y^2+z^2}\,\mathrm{d}z.$$

(a) Show that ω on cube $R = \{(x,y,z) : 0 < x, y, z < \frac{\pi}{4}\}$ is an exact differential form.

(b) Without going back to vector fields, find one $\phi \in C^2$ such that $\omega = \mathrm{d}\phi$ by calculating directly on differential forms.

> **Solution:** (b) $\phi(x,y,z) = x\ln x - x + \tan(x+y) + \frac{1}{2}\ln(y^2+z^2)$.

3. Evaluate the following line integrals of exact differential forms (assumed to be defined in open, connected and simply connected regions). For (a) and (b), work directly on differential forms to simplify the calculation.

(a)

$$\int_{(1,1,1)}^{(2,2,2)} \omega,$$

where $\omega = \frac{1}{y}\,\mathrm{d}x + \left(\frac{1}{z} - \frac{x}{y^2}\right)\,\mathrm{d}y - \frac{y}{z^2}\,\mathrm{d}z$.

(b)

$$\int_{(0,2,1)}^{(1,\frac{\pi}{2},2)} \omega,$$

where $\omega = 2\cos(y)\,\mathrm{d}x + \left(\frac{1}{y} - 2x\sin y\right)\,\mathrm{d}y + \frac{1}{z}\,\mathrm{d}z$.

(c)

$$\int_{(-1,-1,-1)}^{(2,2,2)} \omega,$$

where $\omega = \frac{2x}{x^2+y^2+z^2}\,\mathrm{d}x + \frac{2y}{x^2+y^2+z^2}\,\mathrm{d}y + \frac{2z}{x^2+y^2+z^2}\,\mathrm{d}z$.

Solution: (a)0; (b)$\ln(\frac{\pi}{2})$; (c)$2\ln 2$.

A differential form ω in $\mathbb{R}^3$ is in general given by

$$\omega = M(x,y,z)\,\mathrm{d}x + N(x,y,z)\,\mathrm{d}y + L(x,y,z)\,\mathrm{d}z.$$

When ω is an exact differential form defined in a regular enough region (that is, a conservative vector field defined in a regular enough domain), we have learned how to find the potential function $\phi(x,y,z)$ such that $\omega = \mathrm{d}\phi$ in tutorial 7, which is basically three times of "partial" integrals.

However, there is a class of exact differential forms, given in the form of

$$\begin{aligned}\omega = &\left(M_1(x) + M_2(x,y) + M_3(x,z)\right)\,\mathrm{d}x \\ &+ \left(N_1(y) + N_2(y,x) + N_3(y,z)\right)\,\mathrm{d}y \\ &+ \left(L_1(z) + L_2(z,x) + L_3(z,y)\right)\,\mathrm{d}z,\end{aligned}$$

whose potential functions can be found in a much more easier way. Here is the details:

Step 1: Regroup the terms in ω into

$$\begin{aligned}\omega = &M_2(x,y)\,\mathrm{d}x + N_2(y,x)\,\mathrm{d}y && \text{(the } (x,y) \text{ pair)}\\ &+ N_3(y,z)\,\mathrm{d}y + L_3(z,y)\,\mathrm{d}z && \text{(the } (y,z) \text{ pair)}\\ &+ L_2(z,x)\,\mathrm{d}z + M_3(x,z)\,\mathrm{d}x && \text{(the } (z,x) \text{ pair)}\\ &+ M_1(x)\,\mathrm{d}x \\ &+ N_1(y)\,\mathrm{d}y \\ &+ L_1(z)\,\mathrm{d}z.\end{aligned}$$

Step 2: Write group of terms of j-th row into the form $\mathrm{d}\phi_j$ for some function ϕ_j.(Notice that each row is an exact differential form of dimension no more than 2, a lot easier to calculate, usually solved by trial and error.)

Step 3: Sum up all ϕ_j to obtain one potential function of ω.